

# Quantum mechanics and rational ignorance

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**Abstract.** In the present article we use the quantum formalism to describe the process of choice under rational ignorance. We consider as a basic task a question or an issue where the only answers are 0 and 1. We show that under rational ignorance the opinion state of a person can be described as a qubit state. We analyze the predictions that the quantum formalism give in the study of rational ignorance. We find that answers to different uncorrelated questions are contextual, that is they are influenced by the previous questions, even if uncorrelated. Another interesting prediction is an uncertainty effect which holds when we consider the statistical variance of two or more questions under rational ignorance.

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## 1. Introduction

Is the microscopic world the only part of reality which can be described with the formalism of quantum mechanics? Are there other phenomena which manifest some *quantum features*? In this article, we show that the behavior of people under rational ignorance can be described within the quantum mechanics formalism.

The theory of rational ignorance was first introduced by Downs [1] to study why voters know so little about important issues of politic life, and it is based on the following statement: when the expected benefits of information are too small relative to the costs (for example in an election), people in general choose to remain uninformed. In other words, we have rational ignorance about an issue when any potential benefit deriving from an informed decision about that issue can outweigh the cost of acquiring enough information to make the decision. This has consequences for the decisions made by large numbers of people, such as general elections, where for each person the probability of changing the outcome with one vote is very small.

In this article, we will consider only questions or issues for which the rational ignorance is the dominant process of choice: we also call this situation a rational-ignorance regime. In such a situation, we will show that the quantum formalism can be used to describe simple questions or issues where the only answers are 0 or 1. We will introduce the general formalism of quantum mechanics and in particular some simple notions of quantum information theory, such as the qubit.

The main results of this article are: 1) the opinion-state of a person can be represented, in rational-ignorance regime, by a qubit state; 2) the different issues or questions in rational-ignorance regime can be written as operators acting on the Hilbert space of the states 3) the explicit answer of a person to a question in regime of rational ignorance can be described as a collapse of the opinion vector onto an eigenvector of the corresponding operator 4) there is a contextuality effect, which influences the opinion-state in the case of repeated issues or questions 5) in rational-ignorance regime an uncertainty principle holds, which states that the sum of the variances relevant to at least two questions has a non-trivial lower bound.

It must be noted that our formalism will only describe the behavior under rational ignorance, without considering the psychological motivations that lead to this phenomenon. This article is organized as follows: in section 2 we introduce the basic notation of quantum mechanics and we describe the answers to a question or an issue in rational-ignorance regime in terms of bra and ket. In section 3 we define some important operators which are useful to describe a question, and we introduce the concept of state vector after an answer. In section 4 we write the generalized uncertainty principle for questions under rational-ignorance regime.

## 2. Quantum formalism and basic tests

We briefly introduce the standard bra-ket notation usually used in quantum mechanics, introduced by Dirac [2]. The state of a quantum system is identified with a unit ray in a complex separable Hilbert space, called ket  $|s\rangle$ . When the considered Hilbert space is finite-dimensional, the ket can be written, given an orthonormal basis, as a column vector  $(c_1, c_2, c_3, \dots)^T$ , while the dual vector to the ket, called the bra  $\langle s|$ , can be written, given the same orthonormal basis, as a row vector  $(c_1^*, c_2^*, c_3^*, \dots)$ . The inner product, defined in the Hilbert space, between bra  $\langle s|$  and ket  $|s'\rangle$  can be written as a bracket  $\langle s'|s\rangle$ , thus giving a complex number, called probability amplitude. In finite dimensional Hilbert space,  $|\langle s'|s\rangle|^2$  is the probability to get the state  $|s'\rangle$  as the result of an opportune measure on the state  $|s\rangle$ .

The simplest quantum system is given by a vector in a discrete Hilbert space with dimension 2: this system represents the unit of quantum information, or qubit. Given a basis in the Hilbert space, we can define two vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

representing the quantum analogue to the two possible values 0 and 1 of a classical bit. An important difference is that in the quantum case a state can be in a linear superposition of 0 and 1, that is  $a|0\rangle + b|1\rangle$ , with  $a$  and  $b$  complex numbers.

In quantum mechanics, any observable quantity is described by an hermitian operator (for example,  $\hat{O}$ ). Its eigenvalues  $\{o_i\}$  represent the measurable values of that observable, while its eigenvectors  $\{|o_i\rangle\}$  the corresponding quantum states. In the case of a qubit, there are three important observables, connected with the Pauli matrices:

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of these hermitian operators define three orthonormal basis, with the peculiarity that each eigenvector of one basis is an equal superposition of the eigenvectors of any of the other basis; such basis are also called mutually unbiased:

$$\{|0\rangle, |1\rangle\}, \left\{ \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \right\}, \left\{ \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}} \right\} \quad (1)$$

In the microscopic world, a physical realization of a qubit is provided by a spin 1/2 particle, like an electron. We can measure the spin along the directions  $x$ ,  $y$  and  $z$ , with results  $\pm 1/2$ . These measurements are connected to the observables  $\hat{\sigma}_x$ ,  $\hat{\sigma}_y$ ,  $\hat{\sigma}_z$ .

It is important to recall that two operators  $\hat{A}$  and  $\hat{B}$  are called commuting where  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$ , and that the Pauli matrices are non-commuting operators. In quantum mechanics it is impossible to measure contemporarily two observables with non-commuting operators.

In the context of rational ignorance, what are the observable quantities? As a basic test, we can observe the answer of people to a precise question or issue, in a

situation where the costs (in terms of time and other resources) to get the correct answer outweighs the expected benefits. In particular, we will consider only questions for which the possible answers can only be 0 or 1 (false or true). We can perform on the same person different basic tests, that is we can ask different questions or issues in regime of rational ignorance, in order to know to opinion state of the person.

Given a question or an issue, let us associate to the answers 0 and 1 (false and true respectively) the vectors  $|0\rangle$  and  $|1\rangle$ , corresponding to the truth values of the answer. Since the Pauli matrices have all eigenvalues  $\pm 1/2$ , an important operator is the projector  $\hat{P}_z = (I - \hat{\sigma}_z)/2$ , whose eigenvalues are 0, 1 and whose eigenvectors are  $|0\rangle$  and  $|1\rangle$ . Thus  $\hat{P}_z$  can be considered the hermitian operator (or observable) associated to a question in context of rational ignorance.

Let us now consider a second question, with the important assumption that the answers to question 1 and 2 are statistically independent: for any answer to question 1, there is the same probability that the answer to question 2 is 0 or 1. This means that we must consider the mutually unbiased basis of formula (1), and the second question can be associated only to observable  $\hat{P}_x = (I - \hat{\sigma}_x)/2$  or  $\hat{P}_y = (I - \hat{\sigma}_y)/2$ .

For example in the case of  $\hat{P}_x$ , the answer to question 2 (0 or 1) can be made in correspondence to vectors  $(|0\rangle \pm |1\rangle)/\sqrt{2}$ .

### 3. Quantum collapse and answers

One of the axioms [2] of quantum mechanics states that, given an initial state  $|s\rangle$  in a discrete Hilbert state and an observable  $O$  (with the discrete sets of eigenvalues  $\{o_i\}$  and eigenvectors  $\{|o_i\rangle\}$ ), if we measure one eigenvalue  $o_i$  than the resulting state is  $|o_i\rangle$ . This sudden change of the state vector is called quantum collapse, and its consequences are very important in quantum mechanics.

In the context of rational ignorance, the collapse of the state vector simply states that, given an initial opinion-state  $|s\rangle$  and a question  $\hat{P}_z$ , if the answer is 0 the resulting state is  $|0\rangle$ , while if the answer is 1 the final state is  $|1\rangle$ .

Very simple, isn't it? But let us now give to the same person a second question  $\hat{P}_x$ , again in the context of rational ignorance. Recalling that the inner product between  $|0\rangle$  and  $(|0\rangle \pm |1\rangle)/\sqrt{2}$  is  $1/\sqrt{2}$ , we deduce that answers 0 and 1 for the second question have the same probability  $1/2$ . Thus, when the person answers to the last question, the state vector  $|0\rangle$  is subjected to a second collapse, giving for example  $(|0\rangle - |1\rangle)/\sqrt{2}$  for the answer 0. But if now we ask again the question 1 to the same person, we have  $1/2$  probability to get as answer 1, which is different from before! The rational ignorance situation has manifested an irrational behavior of the person. The question 1 has been asked two times, but what has been changed is the context.

The paradoxical situation is due to the usual belief that we can assign pre-defined elements of reality to individual observables also in regime of rational ignorance. For example, if we ask to a person the three questions associated to observables  $\hat{P}_x$ ,  $\hat{P}_y$ ,  $\hat{P}_z$ , we can write three-elements arrays  $(*, *, *)$ , with  $*$  is 0 or 1, corresponding to

8 the combinations of possible answers. The opinion state of each person can be classically associated to one of these combinations, for example the array  $(1, 0, 1)$ , and any repetition of question  $i$ -th will give the  $i$ -th element of the array, independently from the previous answers. This situation, analogue to classic mechanics, is true when the expected benefits of information are high relative to the costs of getting information. In a rational ignorance regime, instead, we cannot write arrays of answers like  $(1, 0, 1)$  if the related observables are non-commuting. We say that the answers to these questions can not be known contemporarily, thus giving an important limit to the complete knowledge of the opinion state of a person in rational-ignorance regime.

This effect in microscopic world is called quantum contextuality [3], and evidences, for any measurement, the influence of other non-commuting observables previously measured. The repeated sequence of questions in regime of rational ignorance is the analogue of the repeated Stern-Gerlach experiments in quantum mechanics.

#### 4. The uncertainty relations and rational ignorance

In quantum physics, the Heisenberg uncertainty principle is a mathematical limit on the accuracy with which it is possible to measure everything there is to know about a physical system. In its simplest form, it applies to the position and momentum of a single particle in a mono-dimensional continuous space, but more general definitions have given in [4]. In Hilbert spaces with discrete dimension have been formulated other new general forms of uncertainty relations [5].

The uncertainty of an observable  $\hat{A}$  for any given quantum state is defined as the statistical variance of the randomly fluctuating measurement outcomes

$$\delta^2(\hat{A}) = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

Variance zero means that in any experimental measure of the observable we always obtain the same result.

The local uncertainty relations [5] state that, for any set  $\{\hat{A}, \hat{B}, \hat{C}, \dots\}$  of non-commuting operators, there exist a non-trivial limit  $U$  such that  $\delta^2(\hat{A}) + \delta^2(\hat{B}) + \delta^2(\hat{C}) + \dots \geq U$ . In the case of projectors  $\hat{P}_x, \hat{P}_y, \hat{P}_z$  we have

$$\delta^2(\hat{P}_x) + \delta^2(\hat{P}_y) + \delta^2(\hat{P}_z) \geq \frac{1}{2} \quad (2)$$

In terms of rational invariance, formula (2) states that given three uncorrelated questions in regime of rational ignorance, the sum of their statistical variances have a lower non-null bound. We remember that zero variance for one observable means that for every person the answer to the question is the same (complete agreement). Thus equation (2) states that there is a natural limit to the sum of the variances for questions under rational ignorance.

## 5. Conclusions

In the present article, we have given a mathematical description in terms of quantum formalism of questions or issues in situations where the rational ignorance is the dominant process of choice between different answers. The opinion state is thus described in terms of a state vector in a complex Hilbert space. We have used three projection operators, deriving from the Pauli matrices and corresponding to uncorrelated questions: these operators have unbiased basis vectors. We have shown that the hypothesis of rational ignorance leads to a direct influence on the answers to different sequential questions, like the repeated Stern-Gerlach experiments. This example shows that also the opinion state in rational ignorance regime is contextual. A correlated result is finally the uncertainty principle, which evidences a limit also in the statistical knowledge of the opinion state.

This article has the main scope to define a simple formalism to describe rational ignorance, without exploring all the consequences of such an approach. For example, it remains to explore the effects of rational ignorance on the communication process between two persons: we expect to evidence some EPR effects, and to find the concept of entanglement. This study will be presented in a separate paper. Finally, we think that the concept of rational ignorance can be also applied in the study of practical reasoning, such as the association of contexts to words. There are many new papers about this topic, which consider the possible contexts of a word in terms of arrays [6, 7, 8]: in particular a similarity with the quantum mechanics formalism and with the quantum logic is suggested. What is not clearly considered in these papers is a formal description of contextuality effects which the quantum measurement theory evidences, and whose consequences remain to be shown. It is quite paradoxical that in the study of the contexts of a word the problem of contextuality has not yet been fully considered.

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